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FIXED POINT THEOREMS FOR MAPPINGS SATISFYING GENERAL CONTRACTIVE CONDITION OF INTEGRAL TYPE IN G-METRIC SPACES

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Abstract. In this paper, we prove some theorems on fixed and common fixed points for mappings satisfying general contractive condition of integral type in a complete G-metric space. Our results are extensions of the results of Debashis Dey, Anamika Ganguly and Mantu Saha [2] and generalizations of several results in the literature including the results of Branciari [1].
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1. Introduction and Preliminaries

The study of metric fixed point theory has been researched extensively in the past decades, since fixed point theory plays a vital role in Mathematics and applied sciences, such as Optimization, Mathematical models and economic theories. Different authors have generalize the usual notion of metric space (X, d) and extend the known theorems in a more general setting. For example [see (3-7)]. In 2004, Mustafa and Sims [5] introduced a new structure of generalised metric space. This is called the G-metric space (the generalisation of the usual metric space (X, d)). They also introduced new fixed point theorems for various mappings in this new structure.

Definition 1.1. [6] Let X be a non-empty set and $G : X \times X \times X \to R^+$ be a function satisfying the following properties:

(G_1) $G(x, y, z) = 0$ if and only if $x = y = z$
(G_2) $G(x, x, y) > 0$, $\forall x, y \in X$, with $x \neq y$
(G_3) $G(x, x, y) < G(x, y, z)$, $\forall x, y, z \in X$, with $z \neq y$

(G_4) $G(x, y, z) = G(p(x, y, z))$ (symmetry), where p denotes the permutation function.

(G_5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z) \quad \forall a, x, y, z \in X$ (rectangle inequality). Then the function G is called a G-metric on X and the pair (X, G) is called a G-metric space.

Definition 1.2. [6] Let (X, G) be a G-metric space, and $\{x_n\}$ be sequence of points of X. A point $x \in X$ is said to be the limit of the sequence $\{x_n\}$, if $\lim_{n,m\to\infty} G(x, x_n, x_m) = 0$, and we say that the sequence $\{x_n\}$ is G convergent to x. Thus, if $x_n \to x$ in a G-metric space (X, G), then for any $\epsilon > 0$, there exists $n_0 \in N$ such that $G(x, x_n.x_m) < \epsilon$ for all $n, m \geq n_0$.

Definition 1.3. [6] Let (X, G) and (X^1, G^1) be two G-metric spaces, and $f : (X, G) \to (X^1, G^1)$ be a function, then f is said to be G-continuous at a point $x_0 \in X$ if and only if given $\epsilon > 0$, there exists $\delta > 0$ such that $x, y \in X$; $G(x_0, x, y) < \delta$ implies $G^1(f(x_0), f(x), f(y)) < \epsilon$. The function f is G-continuous at X if and only if it is continuous at every points of X.

Definition 1.4. [6] Let (X, G) be a G-metric space, a sequence $\{x_n\}$ is called G-Cauchy if for $\epsilon > 0$, there exists $n_0 \in N$ such that $G(x_n, x_m, x_l) < \epsilon$, for all $m, n, l \geq n_0$, that is, $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.

Definition 1.5. [6] A G-metric space (X, G) is said to be a G-Cauchy complete or complete G-metric space if every G-sequence in (X, G) is G-convergent in (X, G).

Example 1.6. [6] Let $X = [0, 1)$, $T(x) = \frac{x}{4}$ and

$$G(x, y, z) = \max\{|x - y|, |y - z|, |z - x|\}.$$

Then (X, G) is a G-metric space but not G-complete since the sequence

$$x_n = 1 - \frac{1}{n}$$

is G-Cauchy in (X, G) and not G-convergent in (X, G), that is

$$\lim_{n\to\infty}(1 - \frac{1}{n}) = 1 \notin [0, 1).$$

If $X = [0, 1]$, then it is G-complete.

In 2002, Branciari [1] proved the existence of fixed point for a single mapping defined on a complete metric space (X, d) satisfying a general contractive inequality of integral type using the Banach contraction condition.

Theorem 1.7. (Branciari [1]) *Let (X, d) be a complete metric space, $c \in [0, 1)$ and let $f : X \to X$ a mapping such that for $x, y \in X$*

$$\int_0^{d(fx,fy)} \varphi(t)dt \leq c \int_0^{d(x,y)} \varphi(t)dt,$$

where $\varphi : R^+ \to R^+$ is a Lebesgue integrable mapping which is summable, non negative and such that, for each $\epsilon > 0$, $\int_0^\epsilon \varphi(t)dt > 0$. Then f has a unique *fixed point $z \in X$ such that, for each $x \in X$,*

$$\lim_{n\to\infty} f^n x = z.$$

A lot of papers generalizing the results of Branciari [1] for various contractive conditions of integral type have been proved by different authors. Chief among these authors are Rhoades [7] and Dey et al. [2].

Dey et al. [2] proved the following fixed and common fixed point theorems and generalized the results of Branciari [1].

Theorem 1.8. (Dey et al. [2]) *Let f be a self mapping of a complete metric space (X, d) satisfying the following condition:*

$$\int_0^{d(fx,fy)} \varphi(t)dt \le a \int_0^{[d(x,fx)+d(y,fy)]} \varphi(t)dt + b \int_0^{d(x,y)} \varphi(t)dt + c \int_0^{\max\{d(x,fy)+d(y,fx)\}} \varphi(t)dt,$$

for each $x, y \in X$ with nonnegative real numbers a,b,c such that $2a+b+2c < 1$, where $\varphi : R^+ \to R^+$ is a Lebesgue integrable mapping which is summable on each compact subset of R^+, non negative and such that, for each $\epsilon > 0$,

$$\int_0^{\epsilon} \varphi(t)dt > 0.$$

Then f has a unique fixed point $z \in X$ such that, for each $x \in X$,

$$\lim_{n\to\infty} f^n x = z.$$

Theorem 1.9. (Dey et al. [2]) *Let f and g be self mappings of a complete metric space (X, d) satisfying the following condition:*

$$\int_0^{d(fx,gy)} \varphi(t)dt \le a \int_0^{[d(x,fx)+d(y,gy)]} \varphi(t)dt + b \int_0^{d(x,y)} \varphi(t)dt + c \int_0^{\max\{d(x,gy),d(y,fx)\}} \varphi(t)dt,$$

for each $x, y \in X$ with nonnegative real numbers a,b,c such that $2a+b+2c < 1$, where $\varphi : R^+ \to R^+$ is a Lebesgue integrable mapping which is summable on each compact subset of R^+, non negative and such that, for each $\epsilon > 0$,

$$\int_0^{\epsilon} \varphi(t)dt > 0.$$

Then f and g have a unique common fixed point $z \in X$.

The main aim of this paper is to extend the results of Dey et al. [2] to G-metric space and generalize several results in literature including the results of Branciari [1] for a single mapping and then a pair of mappings satisfying a general contractive condition of integral type in a complete generalized metric (ie complete G-metric) space.

2. Main Result

Theorem 2.1. *Let f be a self mapping of a complete G- metric space (X, G) satisfying the following condition:*

$$\int_0^{G(fx,fy,fz)} \varphi(t)dt \leq k \int_0^{G(x,y,z)} \varphi(t)dt + b \int_0^{m(x,y,z)} \varphi(t)dt + c \int_0^{\max\{M(x,y,z)\}} \varphi(t)dt, \quad (2.1)$$

where $m(x, y, z) = [G(x, fx, fx) + G(y, fy, fy) + G(z, fz, fz)]$ and

$$M(x,y,z) = G(x,fy,fy), G(y,fx,fx), G(z,fx,fx), G(x,fz,fz), G(y,fz,fz), G(z,fy,fy),$$

for each $x, y, z \in X$ with nonnegative real numbers a,b,c such that $a+3b+2c < 1$, where $\varphi : R^+ \to R^+$ is a Lebesgue-integrable mapping which is summable (i.e. with finite integral) on each compact subset of R^+, nonnegative and such that

$$\int_0^{x+y} \varphi(t)dt \leq \int_0^{x} \varphi(t)dt + \int_0^{y} \varphi(t)dt$$

for all $x, y \in R^+$ and for each $\epsilon > 0$,

$$\int_0^{\epsilon} \varphi(t)dt > 0. \quad (2.2)$$

Then f has a unique fixed point $p \in X$, $\lim_{n\to\infty} f^n x = z$.

Proof. Let $x_0 \in X$, for simplicity, define $x_n = fx_{n-1}$.

For each integer $n \geq 1$, from (2.1) we get

$$\begin{aligned}\int_0^{G(x_n,x_{n+1},x_{n+1})} \varphi(t)dt = \int_0^{G(fx_{n-1},fx_n,,fx_n)} &\leq a \int_0^{G(x_{n-1},x_n,x_n)} \varphi(t)dt \\ &+b \int_0^{[G(x_{n-1},x_n,x_n)+G(x_n,x_{n+1},x_{n+1})+G(x_n,x_{n+1},x_{n+1})]} \varphi(t)dt \\ &+c \int_0^{\max\{G(x_{n-1},x_{n+1},x_{n+1}),G(x_n,x_n,x_n),G(x_n,x_{n+1},x_{n+1})\}} \varphi(t)dt \\ &= (a+b) \int_0^{G(x_{n-1},x_n,x_n)} \varphi(t)dt + 2b \int_0^{G(x_n,x_{n+1},x_{n+1})} \varphi(t)dt \\ &+c \int_0^{\max\{G(x_{n-1},x_{n+1},x_{n+1}),G(x_n,x_{n+1},x_{n+1})\}} \varphi(t)dt. \quad (2.3)\end{aligned}$$

By G_5, we get

$$G(x_{n-1}, x_{n+1}, x_{n+1}) \leq G(x_{n-1}, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1}). \tag{2.4}$$

But,

$$\begin{aligned}\max\{G(x_{n-1}, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1}), G(x_n, x_{n+1}, x_{n+1})\} \\ = G(x_{n-1}, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1}).\end{aligned} \tag{2.5}$$

Substituting (2.5) in (2.3), we get

$$\begin{aligned}\int_0^{G(x_n,x_{n+1},x_{n+1})} \varphi(t)dt \leq (a+b) \int_0^{G(x_{n-1},x_n,x_n)} \varphi(t)dt \\ +2b \int_0^{G(x_n,x_{n+1},x_{n+1})} \varphi(t)dt \\ +c \int_0^{\max\{G(x_{n-1},x_n,x_n)+G(x_n,x_{n+1},x_{n+1})\}} \varphi(t)dt.\end{aligned} \tag{2.6}$$

Thus,

$$\int_0^{G(x_n,x_{n+1},x_{n+1})} \varphi(t)dt \leq \frac{a+b+c}{1-2b-c} \int_0^{G(x_{n-1},x_n,x_n)} \varphi(t)dt.$$

So,

$$\begin{aligned}\int_0^{G(x_n,x_{n+1},x_{n+1})} \varphi(t)dt &\leq q \int_0^{G(x_{n-1},x_n,x_n)} \varphi(t)dt \\ &\leq q^2 \int_0^{G(x_{n-2},x_{n-1},x_{n-1})} \varphi(t)dt \\ &\vdots \\ &\leq q^n \int_0^{G(x_0,x_1,x_1)} \varphi(t)dt\end{aligned} \tag{2.7}$$

where $\frac{a+b+c}{1-2b-c} = q(say) < 1$.

However, for all $n, m \in N$, $n < m$, we have by repeated use of rectangle inequality in (2.7)

$$\begin{aligned}\int_0^{G(x_n,x_m,x_m)} \varphi(t)dt \leq \int_0^{G(x_n,x_{n+1},x_{n+1})} \varphi(t)dt + \int_0^{G(x_{n+1},x_{n+2},x_{n+2})} \varphi(t)dt \\ + \int_0^{G(x_{n+2},x_{n+3},x_{n+3})} \varphi(t)dt + ... + \int_0^{G(x_{m-1},x_m,x_m)} \varphi(t)dt \\ \leq (q^n + q^{n+1} + ... + q^{m-1}) \int_0^{G(x_0,x_1,x_1)} \leq \frac{q^n}{1-q} \int_0^{G(x_0,x_1,x_1)} \varphi(t)dt.\end{aligned} \tag{2.8}$$

Taking limit of (2.8) as $n, m \to \infty$, we get

$$\lim_{n,m\to\infty} \int_0^{G(x_n,x_m,x_m)} \varphi(t)dt = 0$$

which from (2.2), we get

$$\lim_{n,m\to\infty} G(x_n, x_m, x_m) = 0.$$

For $n, m, l \in N$, (G_5) implies that

$$\int_0^{G(x_n,x_m,x_l)} \varphi(t)dt \leq \int_0^{G(x_n,x_m,x_m)} \varphi(t)dt + \int_0^{G(x_l,x_m,x_m)} \varphi(t)dt.$$

Taking limit as $n, m \to \infty$, we get

$$\lim_{n,m\to\infty} \int_0^{G(x_n,x_m,x_l} \varphi(t)dt = 0$$

which by (2.2), implies that

$$\lim_{n,m\to\infty} G(x_n, x_m, x_l) = 0.$$

This shows that $\{x_n\}$ is a G-Cauchy sequence in X.

By completeness of (X, G), there exists $p \in X$ such that $\{x_n\}$ is G-convergent to p.

Suppose $fp \neq p$, then, from (2.1), we have

$$\begin{aligned}\int_0^{G(x_n,fp,fp)} \varphi(t)dt \leq a \int_0^{G(x_{n-1},p,p)} \varphi(t)dt \\ + b \int_0^{[G(x_{n-1},fp,fp)+G(p,fp,fp)+G(p,fp,fp)]} \varphi(t)dt \\ + c \int_0^{\max\{G(x_{n-1},fp,fp),G(p,x_n,x_n),G(p,fp,fp)\}} \varphi(t)dt.\end{aligned}$$

Taking limit as $n \to \infty$, and using the fact that the function is G-continuous in all its variables, we have

$$\int_0^{G(p,fp,fp)} \varphi(t)dt \leq (3b + c) \int_0^{G(p,fp,fp)} \varphi(t)dt.$$

which by (2.2), we have

$$G(p, fp, fp) \leq (3b + c)G(p, fp, fp),$$

where $3b + c < 1$, which is a contradiction.

So, $p = fp$.

To prove uniqueness, suppose $u \neq p$, such that $fu = u$, then by (2.1),

$$\begin{aligned}\int_0^{G(p,u,u)} \varphi(t)dt \leq a &\int_0^{G(p,u,u)} \varphi(t)dt \\ &+ b \int_0^{[G(p,p,p)+G(u,u,u)+G(u,u,u)]} \varphi(t)dt \\ &+ c \int_0^{\max\{G(p,u,u),G(u,p,p),G(p,p,p)\}} \varphi(t)dt.\end{aligned}$$

Thus,

$$\int_0^{G(p,u,u)} \varphi(t)dt \leq (a+c) \int_0^{G(p,u,u)} \varphi(t)dt. \tag{2.9}$$

where $G(u,p,p) = G(p,u,u)$. (Symmetry)
Since $a + c < 1$, it implies that

$$\int_0^{G(p,u,u)} \varphi(t)dt = 0$$

which by (2.2), implies that $G(p,u,u) = 0$ or $p = u$, so the fixed point is unique. This ends the proof.

Remark 2.2. By setting $\varphi(t) = 1$ over R^+, every contractive condition of integral type transform to corresponding contractive condition not involving integrals.

Remark 2.3. (i) Our Theorem 2.1 give the results of Branciari [1] if $b = c = 0$ in (2.1).

(ii) Our Theorem 2.1 also extend the main results (Theorem 2.1 and Theorem 2.3 of Dey et al.[2]) to G-metric space. The following example verifies Theorem 2.1.

Next, we extend our result to a pair of mappings.

Theorem 2.4. *Let f and g be self mappings of a complete G- metric space (X,G) satisfying the following condition:*

$$\begin{aligned}\int_0^{G(fx,gy,gz)} \varphi(t)dt \leq k &\int_0^{G(x,y,z)} \varphi(t)dt + b \int_0^{m(x,y,z)} \varphi(t)dt \\ &+ c \int_0^{\max\{M(x,y,z)\}} \varphi(t)dt,\end{aligned} \tag{2.10}$$

where $m(x,y,z) = [G(x,fx,fx) + G(y,gy,gy) + G(z,gz,gz)]$ and

$$\begin{aligned}M(x,y,z) =& G(x,gy,gy), G(y,fx,fx), G(z,fx,fx),\\ &G(x,gz,gz), G(y,gz,gz), G(z,gy,gy),\end{aligned}$$

for each $x, y, z \in X$ with nonnegative real numbers a,b,c such that $a+3b+2c < 1$, where $\varphi : R^+ \to R^+$ is a Lebesgue- integrable mapping which is summable *(i.e. with finite integral) on each compact subset of* R^+*, nonnegative and such that for*

$$\int_0^{x+y} \varphi(t)dt \leq \int_0^{x} \varphi(t)dt + \int_0^{y} \varphi(t)dt$$

and for each $\epsilon > 0$,

$$\int_0^{\epsilon} \varphi(t)dt > 0. \tag{2.11}$$

Then f *and* g *have a unique common fixed point* $p \in X$.

Proof. Let $x_0 \in X$, for simplicity, define $x_{2n+1} = fx_2n$ and $x_{2n+2} = gx_{2n+1}$.

For each integer $n \geq 0$, from (2.10) we get

$$\begin{aligned}
&\int_0^{G(x_{2n+1},x_{2n+2},x_{2n+2})} \varphi(t)dt = \int_0^{G(fx_{2n},gx_{2n+1},,gx_{2n+1})} \\
&\leq a\int_0^{G(x_{2n},x_{2n+1},x_{2n+1})} \varphi(t)dt + b\int_0^{J} \varphi(t)dt + c\int_0^{\max\{L\}} \varphi(t)dt \\
&= (a+b)\int_0^{G(x_{2n},x_{2n+1},x_{2n+1})} \varphi(t)dt + 2b\int_0^{G(x_{2n+1},x_{2n+2},x_{2n+2})} \varphi(t)dt \\
&\quad +c\int_0^{\max\{G(x_{2n},x_{2n+2},x_{2n+2}),G(x_{2n+1},x_{2n+2},x_{2n+2})\}} \varphi(t)dt.
\end{aligned} \tag{2.12}$$

where

$$J = [G(x_{2n}, x_{2n+1}, x_{2n+1}) + G(x_{2n+1}, x_{2n+2}, x_{2n+2}) + G(x_{2n+1}, x_{2n+2}, x_{2n+2})]$$

$$L = G(x_{2n}, x_{2n+2}, x_{2n+2}), G(x_{2n+1}, x_{2n+1}, x_{2n+1}), G(x_{2n+1}, x_{2n+2}, x_{2n+2}).$$

By G_5, we get

$$G(x_{2n}, x_{2n+2}, x_{2n+2}) \leq G(x_2n, x_{2n+1}, x_{2n+1})+G(x_{2n+1}, x_{2n+2}, x_{2n+2}). \tag{2.13}$$

But,

$$\begin{aligned}
&\max\{G(x_{2n}, x_{2n+1}, x_{2n+1}) + G(x_{2n+1}, x_{2n+2}, x_{2n+2}), G(x_{2n+1}, x_{2n+2}, x_{2n+2})\} \\
&\qquad = G(x_{2n}, x_{2n+1}, x_{2n+1}) + G(x_{2n+1}, x_{2n+2}, x_{2n+2}).
\end{aligned} \tag{2.14}$$

Substituting (2.14) in (2.12), we get

$$\begin{aligned}
&\int_0^{G(x_{2n+1},x_{2n+2},x_{2n+2})} \varphi(t)dt \leq (a+b)\int_0^{G(x_{2n},x_{2n+1},x_{2n+1})} \varphi(t)dt \\
&+2b\int_0^{G(x_{2n+1},x_{2n+2},x_{2n+2})} \varphi(t)dt + c\int_0^{\max\{T(x,y,z)\}} \varphi(t)dt.
\end{aligned} \tag{2.15}$$

where $T(x, y, z) = G(x_{2n}, x_{2n+1}, x_{2n+1}) + G(x_{2n+1}, x_{2n+2}, x_{2n+2})$.

Thus,

$$\int_0^{G(x_{2n+1},x_{2n+2},x_{2n+2})} \varphi(t)dt \leq \frac{a+b+c}{1-2b-c}\int_0^{G(x_{2n},x_{2n+1},x_{2n+1})} \varphi(t)dt.$$

So,

$$\begin{aligned}\int_0^{G(x_{2n+1},x_{2n+2},x_{2n+2})}\varphi(t)dt &\leq q\int_0^{G(x_{2n},x_{2n+1},x_{2n+1})}\varphi(t)dt\\ &\leq q^2\int_0^{G(x_{2n-1},x_{2n},x_{2n})}\varphi(t)dt\\ &\vdots\\ &\leq q^n\int_0^{G(x_0,x_1,x_1)}\varphi(t)dt \qquad (2.16)\end{aligned}$$

where $\frac{a+b+c}{1-2b-c} = q(say) < 1$.

We now show that $\{x_n\}$ is a G-Cauchy sequence.

For all $n, m \in N$, $n < m$, we have by repeated use of rectangle inequality in (2.16)

$$\begin{aligned}\int_0^{G(x_{2n},x_{2m},x_{2m})}\varphi(t)dt &\leq \int_0^{G(x_{2n},x_{2n+1},x_{2n+1})}\varphi(t)dt\\ &+\int_0^{G(x_{2n+1},x_{2n+2},x_{2n+2})}\varphi(t)dt\\ &+\int_0^{G(x_{2n+2},x_{2n+3},x_{2n+3})}\varphi(t)dt + ... + \int_0^{G(x_{2m-1},x_{2m},x_{2m})}\varphi(t)dt\\ &\leq (q^{2n}+q^{2n+1}+...+q^{2m-1})\int_0^{G(x_0,x_1,x_1)}\\ &\leq \frac{q^{2n}}{1-q}\int_0^{G(x_0,x_1,x_1)}\varphi(t)dt. \qquad (2.17)\end{aligned}$$

Taking limit of (2.17) as $n, m \to \infty$, we get

$$\lim_{n,m\to\infty}\int_0^{G(x_{2n},x_{2m},x_{2m})}\varphi(t)dt = 0$$

which from (2.11), we get

$$\lim_{n,m\to\infty} G(x_{2n}, x_{2m}, x_{2m}) = 0.$$

For $n, m, l \in N$, (G_5) implies that

$$\int_0^{G(x_{2n},x_{2m},x_{2l})}\varphi(t)dt \quad \leq \quad \int_0^{G(x_{2n},x_{2m},x_{2m})}\varphi(t)dt + \int_0^{G(x_{2l},x_{2m},x_{2m})}\varphi(t)dt$$

Taking limit as $n, m \to \infty$, we get

$$\lim_{n,m\to\infty}\int_0^{G(x_{2n},x_{2m},x_{2l})}\varphi(t)dt = 0$$

which by (2.11), implies that

$$\lim_{n,m\to\infty} G(x_{2n}, x_{2m}, x_{2l}) = 0.$$

This shows that $\{x_n\}$ is a G-Cauchy sequence in X.

By completeness of (X, G), there exists $p \in X$ such that $\{x_n\}$ is G-convergent to p.

From (2.10), we get

$$\int_0^{G(fp,x_{2n+2},x_{2n+2})} \varphi(t)dt = \int_0^{G(fp,gx_{2n+1},,gx_{2n+1})} \leq a \int_0^{G(p,x_{2n+1},x_{2n+1})} \varphi(t)dt$$

$$+b \int_0^{[G(p,fp,fp)+G(x_{2n+1},x_{2n+2},x_{2n+2})+G(x_{2n+1},x_{2n+2},x_{2n+2})]} \varphi(t)dt$$

$$+c \int_0^{\max\{G(p,x_{2n+2},x_{2n+2}),G(x_{2n+1},fp,fp),G(x_{2n+1},x_{2n+2},x_{2n+2})\}} \varphi(t)dt.$$

Taking limit as $n, m \to \infty$, we get

$$\begin{aligned}\int_0^{G(fp,p,p)} \varphi(t)dt \leq a &\int_0^{G(p,p,p)} \varphi(t)dt \\ &+ b \int_0^{[G(p,fp,fp)+G(p,p,p)+G(p,p,p)]} \varphi(t)dt \\ &+ c \int_0^{\max\{G(p,p,p),G(p,fp,fp)\}} \varphi(t)dt\end{aligned}$$

Thus,

$$\int_0^{G(fp,p,p)} \varphi(t)dt \quad \leq \quad (b+c) \int_0^{G(p,fp,fp)} \varphi(t)dt$$

As $b + c < 1$, we have

$$\int_0^{G(fp,p,p)} \varphi(t)dt = 0$$

which from (2.11), implies that

$$G(fp, p, p) = 0.$$

Similarly, it can be shown that $gp = p$.

We now show that p is the unique common fixed point of f and g.

Suppose it is not, let v be another common fixed point of f and g, then, from (2.10), we have

$$\begin{aligned}\int_0^{G(p,v,v)} \varphi(t)dt &= \int_0^{G(fp,gv,,gv)} \\ &\leq a \int_0^{G(p,v,v)} \varphi(t)dt \\ &+ b \int_0^{[G(p,fp,fp)+G(v,gv,gv)+G(v,gv,gv)]} \varphi(t)dt \\ &+ c \int_0^{\max\{G(p,gv,gv),G(v,fp,fp),G(v,gv,gv)\}} \varphi(t)dt \\ &\leq (a+c) \int_0^{G(p,v,v)} \varphi(t)dt,\end{aligned}$$

where $G(v, fp, fp) = G(v, p, p)$ (Since $fp = p$)
$G(p, gv, gv) = G(p, v, v)$ (Since $gv = v$) and
$G(v, p, p) = G(p, v, v)$ (Symmetric property in G_3)
Since $a + c < 1$, it implies that

$$\int_0^{G(p,v,v)} \varphi(t)dt = 0$$

which by (2.11), implies that

$$G(p, v, v) = 0$$

or $p = v$, so the common fixed point is unique. This ends the proof.

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